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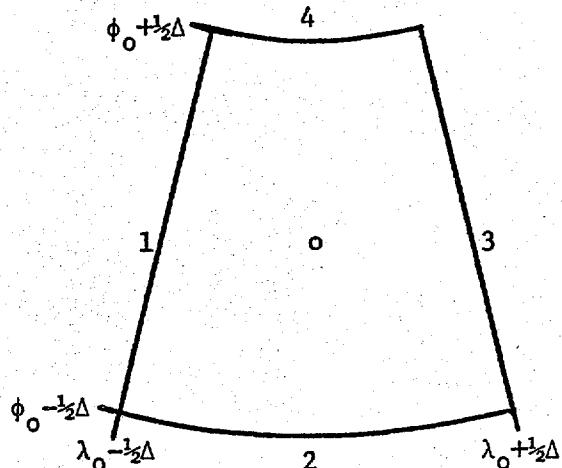
Difference Methods in Spherical Coordinates to Conserve Mass

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Symbols

λ, ϕ	longitude, latitude
σ	vertical coordinate
u, v	eastward, northward
	velocity components
D	horizontal divergence
p	pressure
r	earth's radius
Δ	interval of angle in regular longitude-latitude grid
A	area of mesh element shown at right



The area of the areal element is

$$\begin{aligned}
 A(\phi_0) &= \int_{\lambda_1}^{\lambda_3} \int_{\phi_2}^{\phi_4} (r d\phi) (r \cos \phi d\lambda) \\
 &= r^2 \Delta [\sin \phi]_{\phi_2}^{\phi_4} \\
 &= r^2 \Delta^2 (\sin \phi)_{\phi_0} \\
 &= r^2 \Delta^2 \cos \phi_0 \cdot \frac{\sin \frac{1}{2}\Delta}{\frac{1}{2}\Delta}
 \end{aligned}$$

Consider horizontal divergence:

$$\begin{aligned}
 D &= \frac{\partial u}{r \cos \phi \partial \lambda} + \frac{\partial v}{r \partial \phi} - \frac{v \sin \phi}{r \cos \phi} \\
 &= \frac{1}{r \cos \phi} \left[\frac{\partial u}{\partial \lambda} + \frac{\partial (v \cos \phi)}{\partial \phi} \right]
 \end{aligned}$$

The average divergence, \bar{D} , over the areal element is

$$\begin{aligned}
 \bar{D} &= \frac{1}{A} \int_{\lambda_1}^{\lambda_3} \int_{\phi_2}^{\phi_4} D (r d\phi) (r \cos \phi d\lambda) \\
 &= \frac{r}{A} \int_{\lambda_1}^{\lambda_3} \int_{\phi_2}^{\phi_4} \left[\frac{\partial u}{\partial \lambda} + \frac{\partial (v \cos \phi)}{\partial \phi} \right] d\phi d\lambda \\
 &= \frac{r}{A} \int_{\phi_2}^{\phi_4} (u_3 - u_1) d\phi + \frac{r}{A} \int_{\lambda_1}^{\lambda_3} (v_4 \cos \phi_4 - v_2 \cos \phi_2) d\lambda \\
 &= \frac{r\Delta}{A} \left[\int_{\phi_2}^{\phi_4} u_\lambda d\phi + \int_{\lambda_1}^{\lambda_3} (v \cos \phi)_\phi d\lambda \right]
 \end{aligned}$$

I now assume u_3 and u_1 to be linear with respect to ϕ , and v_4 and v_2 to be linear with respect to λ . Then

$$\begin{aligned}\bar{D} &= \frac{r\Delta^2}{A} \left[\bar{u}_\lambda^\phi + (\bar{v}_\lambda^\lambda \cos \phi)_{\phi} \right] \quad (1) \\ &= \frac{1}{r(\sin \phi)} \left[\bar{u}_\lambda^\phi + \bar{v}_\phi^\lambda \overline{(\cos \phi)}^\phi + \bar{v}_\lambda^\lambda \phi (\cos \phi)_{\phi} \right]\end{aligned}$$

Note that

$$(\sin \phi)_\phi = S \cos \phi_0$$

$$(\cos \phi)_\phi = -S \sin \phi_0$$

$$\overline{(\cos \phi)}^\phi = \cos \frac{1}{2}\Delta \cdot \cos \phi_0$$

where $S(\Delta) \equiv \frac{\sin \frac{1}{2}\Delta}{\frac{1}{2}\Delta}$

Let $T(\Delta) = \frac{\tan \frac{1}{2}\Delta}{\frac{1}{2}\Delta}$

Then $\bar{D} = \frac{1}{S} \cdot \frac{\bar{u}_\lambda^\phi}{r \cos \phi_0} + \frac{1}{T} \frac{\bar{v}_\phi^\lambda}{r} - \frac{\bar{v}_\lambda^\lambda \phi \sin \phi_0}{r \cos \phi_0} \quad (2)$

In (1) and (2), if the wind vector, \vec{V} , is replaced by the vector, $p_\sigma \vec{V}$, then

$$\begin{aligned}\bar{v} \cdot p_\sigma \vec{v} &= \frac{1}{r(\sin \phi)} \left[(\bar{p}_\sigma u)_\lambda^\phi + (\bar{p}_\sigma v)^\lambda \cos \phi \right] \\ &= \frac{(\bar{p}_\sigma u)_\lambda^\phi}{Sr \cos \phi_0} + \frac{(\bar{p}_\sigma v)^\lambda}{Tr} - \frac{\bar{p}_\sigma v \sin \phi_0}{r \cos \phi_0} \\ &= \frac{\bar{u}_\lambda^\phi}{Sr \cos \phi_0} + \frac{\bar{v}_\phi^\lambda}{Tr} \\ &\quad + \frac{\bar{p}_\sigma u_\lambda^\phi}{Sr \cos \phi_0} + \frac{\bar{p}_\sigma v_\phi^\lambda}{Tr} - \frac{\bar{p}_\sigma v \sin \phi_0}{r \cos \phi_0}\end{aligned}$$

The table below shows S and T for various values of Δ .

Δ	S	T
1°	.999 9873	1.000 0254
1.5°	.999 9714	1.000 0571
2°	.999 9492	1.000 1016
2.5°	.999 9207	1.000 1587
3°	.999 8858	1.000 2285
3.75°	.999 8215	1.000 3571
5°	.999 6827	1.000 6351
10°	.998 7312	1.002 5462

Although the "correction" factors are close to unity, without the correction an error could conceivably grow exponentially with time to something substantial. I would argue that in any case they should be included in the continuity equation, because the theory is sound, and the cost of their inclusion is minimal.